(I) Relativistic Motions around a Black Hole

$$\int dt\,\mathcal{L} = -\int dt\, mc^2\sqrt{f(r) - \frac{\dot{r}^2}{c^2 f(r)} - r^2\frac{\dot{\theta}^2 + (\sin\theta)^2\dot{\phi}^2}{c^2}}$$

(r, θ, ϕ): spherical coordinate of R^3

$$f(r) = 1 - \frac{2G_N M}{c^2 r}$$

1) identify all constants of motion

2) classify trajectories outside the horizon $r > \frac{2G_N M}{c^2}$

3) classify trajectories inside the horizon $r < \frac{2G_N M}{c^2}$

(II) Relativistic Motions around an Extremally Charged Black Hole

$$\int dt\,\mathcal{L} = -\int dt\, mc^2 \sqrt{\tilde{f}(r) - \frac{\dot{r}^2}{c^2\tilde{f}(r)} - r^2\frac{\dot{\theta}^2 + (\sin\theta)^2\dot{\phi}^2}{c^2}}$$

(r, θ, ϕ): spherical coordinate of R^3

$$\tilde{f}(r) = \left(1 - \frac{G_N M}{c^2 r}\right)^2$$

1) classify trajectories outside the horizon $r > \frac{G_N M}{c^2}$

2) classify trajectories inside the horizon $r < \frac{G_N M}{c^2}$

3) trajectories inside the horizon are qualitatively different from those of case (I); describe typical trajectories and discuss their eventual fate; keep in mind that once an object fall into the horizon, it cannot come out again through the same horizon.